8-5 The First Derivative Test

1. a) $f(x) = 3x^2 - 4x + 13$

$f'(x) = 6x - 4$

$= 2(3x-2)$

$3x - 2 = 0$

$3x = 2$

$x = \frac{2}{3}$

	$3x-2$	$f'(x)$	
$x < \frac{2}{3}$	$-$	$-$	decr
$x > \frac{2}{3}$	$+$	$+$	incr

changes from neg to pos so its a min. at $f(\frac{2}{3}) = \frac{35}{3}$

$f(\frac{2}{3}) = 3(\frac{2}{3})^2 - 4(\frac{2}{3}) + 13$

$= \frac{12}{9} - \frac{8}{3} + 13$

$= \frac{12}{9} - \frac{24}{9} + \frac{117}{9}$

$= \frac{105}{9} = \frac{35}{3}$

b) $f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12$

$= 3(x^2 - 4)$

$= 3(x-2)(x+2)$

$x - 2 = 0$ or $x + 2 = 0$

$x = 2$ $\quad$ $x = -2$

	$x-2$	$x+2$	$f'(x)$	
$x < -2$	$-$	$-$	$+$	incr.
$-2 < x < 2$	$-$	$+$	$-$	decr
$x > 2$	$+$	$+$	$+$	incr.

-2 is a max.

2 is a min.

$f(2) = 2^3 - 12 \cdot 2 - 5$

$= 8 - 24 - 5$

$= -21$

$f(-2) = (-2)^3 - 12(-2) - 5$

$= -8 + 24 - 5$

$= 11$

local max: $f(-2) = 11$

local min: $f(2) = -21$

8-5 cont.

c) $g(x) = 2 + 5x - x^5$

$g'(x) = 5 - 5x^4$

$= 5(1 - x^4)$

$= 5(1 - x^2)(1 + x^2)$

$= 5(1 - x)(1 + x)(1 + x^2)$

$1 - x = 0$, $1 = x$; $1 + x = 0$, $x = -1$; $1 + x^2 = 0$, $x^2 = -1$ no solution

	$1-x$	$1+x$	$1+x^2$	$g'(x)$
$x < -1$	+	−	+	−
$-1 < x < 1$	+	+	+	+
$x > 1$	−	+	+	−

min @ −1

max @ 1

$g(-1) = 2 + 5(-1) - (-1)^5$
$= 2 - 5 + 1$
$= -2$

$g(1) = 2 + 5(1) - 1^5$
$= 2 + 5 - 1$
$= 6$

loc. min.: $g(-1) = -2$ loc. max.: $g(1) = 6$

d) $y = x^4 - x^3$

$y' = 4x^3 - 3x^2$

$= x^2(4x - 3)$

$x^2 = 0$, $x = 0$; $4x - 3 = 0$, $4x = 3$, $x = \frac{3}{4}$

	x^2	$4x-3$	y'
$x < 0$	+	−	−
$0 < x < \frac{3}{4}$	+	−	−
$x > \frac{3}{4}$	+	+	+

min @ $\frac{3}{4}$

$y = \left(\frac{3}{4}\right)^4 - \left(\frac{3}{4}\right)^3$
$= \frac{81}{256} - \frac{27}{64}$
$= \frac{81}{256} - \frac{108}{256}$
$= -\frac{27}{256}$

loc. min.: $\frac{-27}{256}$ when $x = \frac{3}{4}$

8-5 cont.

2. a) $f(x) = 2 + 6x - 6x^2$

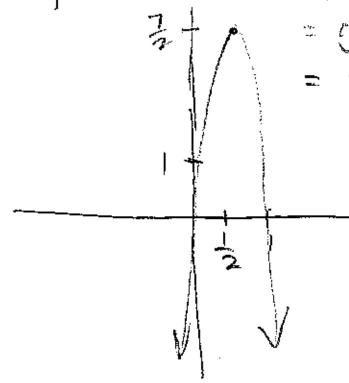
$f'(x) = 6 - 12x$
$= 6(1 - 2x)$

$1 - 2x = 0$
$1 = 2x$
$\frac{1}{2} = x$ ← critical number

	$1-2x$	$f'(x)$	interval
$x < \frac{1}{2}$	+	+	incr. $(-\infty, \frac{1}{2})$
$x > \frac{1}{2}$	−	−	decr. $(\frac{1}{2}, \infty)$

$f(\frac{1}{2}) = 2 + 6(\frac{1}{2}) - 6(\frac{1}{2})^2$
$= 2 + 3 - \frac{6}{4}$
$= 5 - \frac{3}{2}$
$= 3\frac{1}{2}$ or $\frac{7}{2}$

loc. max @ $x = \frac{1}{2}$
$f(\frac{1}{2}) = \frac{7}{2}$



b) $g(x) = 1 + 3x^2 - 2x^3$

$g'(x) = 6x - 6x^2$
$= 6x(1 - x)$

$6x = 0$, $1 - x = 0$
$x = 0$, $1 = x$ critical numbers

	$6x$	$1-x$	$g'(x)$	interval
$x < 0$	−	+	−	decr. $(-\infty, 0)$
$0 < x < 1$	+	+	+	incr. $(0, 1)$
$x > 1$	+	−	−	decr. $(1, \infty)$

$f(0) = 1 + 3 \cdot 0^2 - 2 \cdot 0^3$
$= 1 + 0 - 0$
$= 1$

$f(1) = 1 + 3(1)^2 - 2(1)^3$
$= 1 + 3 - 2$
$= 2$

loc min: $f(0) = 1$
loc max: $f(1) = 2$

8-5 cont.

2. c) $g(x) = 3x^4 - 16x^3 + 18x^2 + 1$

$g'(x) = 12x^3 - 48x^2 + 36x$
$= 12x(x^2 - 4x + 3)$
$= 12x(x-3)(x-1)$

$12x = 0,\ x-3=0,\ x-1=0$
$x=0 \quad x=3 \quad x=1$
critical numbers

	$12x$	$x-3$	$x-1$	$g'(x)$	interval
$x<0$	$-$	$-$	$-$	$-$	decr. $(-\infty, 0)$
$0<x<1$	$+$	$-$	$-$	$+$	incr. $(0, 1)$
$1<x<3$	$+$	$-$	$+$	$-$	decr. $(1, 3)$
$x>3$	$+$	$+$	$+$	$+$	incr. $(3, \infty)$

$f(0) = 3(0)^4 - 16(0)^3 + 18(0)^2 + 1$
$= 0 - 0 - 0 + 1$
$= 1$

$f(3) = 3(3)^4 - 16(3)^3 + 18(3)^2 + 1$
$= 243 - 432 + 162 + 1$
$= -26$

$f(1) = 3(1)^4 - 16(1)^3 + 18(1)^2 + 1$
$= 3 - 16 + 18 + 1$
$= 6$

loc. min: $f(0) = 1$, $f(3) = -26$
loc. max: $f(1) = 6$

d) $h(x) = 3x^5 - 5x^3$

$h'(x) = 15x^4 - 15x^2$
$= 15x^2(x^2 - 1)$
$= 15x^2(x-1)(x+1)$

$15x^2 = 0,\ x-1=0,\ x+1=0$
$x=0 \quad x=1 \quad x=-1$
critical numbers

see next page

8-5 cont.

2. d) cont.

	$15x^2$	$x-1$	$x+1$	$h'(x)$	interval
$x<-1$	+	−	−	+	incr $(-\infty,-1)$
$-1<x<0$	+	−	+	−	decr $(-1,1)$
$0<x<1$	+	−	+	−	
$x>1$	+	+	+	+	incr $(1,\infty)$

$h(0) = 3(0)^5 - 5(0)^3$
$= 0 - 0$
$= 0$

$h(-1) = 3(-1)^5 - 5(-1)^3$
$= -3 + 5$
$= 2$

$h(1) = 3(1)^5 - 5(1)^3$
$= 3 - 5$
$= -2$

loc. min: $h(1) = -2$
loc. max: $h(-1) = 2$

3. a) $f(x) = 27 + x - x^2$
$f'(x) = 1 - 2x$

$1 - 2x = 0$
$1 = 2x$
$\frac{1}{2} = x$

	$1-2x$	$f'(x)$	interval
$x<\frac{1}{2}$	+	+	incr.
$x>\frac{1}{2}$	−	−	decr.

$f(\frac{1}{2}) = 27 + \frac{1}{2} - (\frac{1}{2})^2$
$= 27 + \frac{1}{2} - \frac{1}{4}$
$= \frac{108}{4} + \frac{2}{4} - \frac{1}{4}$
$= \frac{109}{4}$

Because there is only 1 critical point and it switches from incr → dec, it must be an abs. max

abs. max.: $f(\frac{1}{2}) = \frac{109}{4}$

8-5 cont.

3. b) $f(x) = 3 - \frac{1}{\sqrt{x^2+1}}$

$= 3 - (x^2+1)^{-\frac{1}{2}}$

$f'(x) = \cancel{\frac{1}{2}}(x^2+1)^{-\frac{3}{2}} \cdot \cancel{2}x$

$= \frac{x}{(x^2+1)^{\frac{3}{2}}}$

When $x > 0$, then $f'(x) > 0 \rightarrow$ incr.
When $x < 0$, then $f'(x) < 0 \rightarrow$ decr.

$f(0) = 3 - \frac{1}{\sqrt{0^2+1}}$
$= 3 - \frac{1}{1}$
$= 2$

Abs. max. : $f(0) = 2$

c) $g(x) = \frac{x^2-1}{x^2+1}$

$g'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$

$= \frac{\cancel{2x^3} + 2x - \cancel{2x^3} + 2x}{(x^2+1)^2}$

$= \frac{4x}{(x^2+1)^2}$

when $x > 0$ then $g'(x) > 0 \rightarrow$ incr.
when $x < 0$ then $g'(x) < 0 \rightarrow$ decr.

Abs. max. : $g(0) = -1$

$g(0) = \frac{0^2-1}{0^2+1}$
$= \frac{-1}{1}$
$= -1$

d) $g(x) = \frac{x^2-x+1}{x^2+1}$

$g'(x) = \frac{(x^2+1)(2x-1) - (x^2-x+1)(2x)}{(x^2+1)^2}$

$= \frac{2x^3 - x^2 + 2x - 1 - (2x^3 - 2x^2 + 2x)}{(x^2+1)^2}$

$= \frac{\cancel{2x^3} - x^2 \cancel{+2x} - 1 \cancel{-2x^3} + 2x^2 \cancel{-2x}}{(x^2+1)^2}$

$= \frac{x^2-1}{(x^2+1)^2}$

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8-5 cont.

3. d) cont.

when $x > 1$ then $g'(x) > 0 \rightarrow$ incr.
when $x < 1$ then $g'(x) < 0 \rightarrow$ decr.

$$g(1) = \frac{1^2 - 1 + 1}{1^2 + 1} = \frac{1 - 1 + 1}{1 + 1} = \frac{1}{2}$$

Abs. min : $g(1) = \frac{1}{2}$