9-5 The Second Derivative Test

1. a) $f(x) = 3x^2 - 4x + 13$

$f'(x) = 6x - 4$

$0 = 6x - 4$

$4 = 6x$

$\frac{4}{6} = x$

$\frac{2}{3} = x$

$f''(x) = 6$

$f'(\frac{2}{3}) = 0$

$f''(\frac{2}{3}) = 6$ pos

$f(\frac{2}{3}) = 3(\frac{2}{3})^2 - 4(\frac{2}{3}) + 13$

$= \frac{4}{3} - \frac{8}{3} + \frac{39}{3}$

$= \frac{35}{3}$

local min is $\frac{35}{3}$

b) $g(x) = 2x^3 - 48x - 17$

$g'(x) = 6x^2 - 48$

$0 = 6x^2 - 48$

$48 = 6x^2$

$8 = x^2$

$\pm\sqrt{8} = x$

$\pm 2\sqrt{2} = x$

$f''(x) = 12x$

$f''(2\sqrt{2}) = 12(2\sqrt{2})$

$= 24\sqrt{2}$ pos

$f''(-2\sqrt{2}) = 12(-2\sqrt{2})$

$= -24\sqrt{2}$ neg

$f(2\sqrt{2}) = 2(2\sqrt{2})^3 - 48(2\sqrt{2}) - 17$

$= 32\sqrt{2} - 96\sqrt{2} - 17$

$= -64\sqrt{2} - 17$ local max.

$f(-2\sqrt{2}) = 2(-2\sqrt{2})^3 - 48(-2\sqrt{2}) - 17$

$= -32\sqrt{2} + 96\sqrt{2} - 17$

$= 64\sqrt{2} - 17$ local min

c) $h(x) = x^3 - 9x^2 + 24x - 10$

$h'(x) = 3x^2 - 18x + 24$

$0 = 3x^2 - 18x + 24$

$0 = x^2 - 6x + 8$

$0 = (x-2)(x-4)$

$x - 2 = 0$ or $x - 4 = 0$

$x = 2$ $\quad$ $x = 4$

$f''(x) = 6x - 18$

$f''(2) = 6 \cdot 2 - 18$

$= 12 - 18$

$= -6$ neg.

$f''(4) = 6 \cdot 4 - 18$

$= 24 - 18$

$= 6$ pos

$f(2) = (2)^3 - 9(2)^2 + 24(2) - 10$

$= 8 - 36 + 48 - 10$

$= 10$

local max

$f(4) = (4)^3 - 9(4)^2 + 24(4) - 10$

$= 64 - 144 + 96 - 10$

$= 6$

local min

9-5 cont.

1. d) $F(x) = 3x^4 - 16x^3 + 18x^2 + 1$

$F'(x) = 12x^3 - 48x^2 + 36x$

$0 = 12x^3 - 48x^2 + 36x$

$0 = 12x(x^2 - 4x + 3)$

$0 = 12x(x-3)(x-1)$

$12x = 0$, $x - 3 = 0$, $x - 1 = 0$

$x = 0$, $x = 3$, $x = 1$

$F(0) = 3 \cdot 0^4 - 16 \cdot 0^3 + 18 \cdot 0^2 + 1$

$= 0 - 0 + 0 + 1$

$= 1$ local min

$F(3) = 3 \cdot 3^4 - 16 \cdot 3^3 + 18 \cdot 3^2 + 1$

$= 243 - 432 + 162 + 1$

$= -26$ local min

$F''(x) = 36x^2 - 96x + 36$

$F''(0) = 36 \cdot 0^2 - 96 \cdot 0 + 36$

$= 0 - 0 + 36$

$= 36$ pos

$F''(3) = 36 \cdot 3^2 - 96 \cdot 3 + 36$

$= 324 - 288 + 36$

$= 72$ pos

$F''(1) = 36 \cdot 1^2 - 96 \cdot 1 + 36$

$= 36 - 96 + 36$

$= -24$ neg.

$F(1) = 3 \cdot 1^4 - 16 \cdot 1^3 + 18 \cdot 1^2 + 1$

$= 3 - 16 + 18 + 1$

$= 6$ local max

e) $G(x) = x^2 + \frac{16}{x}$

$= x^2 + 16x^{-1}$

$G'(x) = 2x - 16x^{-2}$

$= 2x - \frac{16}{x^2}$

$= \frac{2x^3}{x^2} - \frac{16}{x^2}$

$= \frac{2x^3 - 16}{x^2}$

$0 = \frac{2x^3 - 16}{x^2}$

$0 = 2x^3 - 16$

$16 = 2x^3$

$8 = x^3$

$2 = x$

$G''(x) = 2 + 32x^{-3}$

$= 2 + \frac{32}{x^3}$

$G(2) = 2 + \frac{32}{2^3}$

$= 2 + \frac{32}{8}$

$= 2 + 4$

$= 6$ pos.

$G(6) = 2^2 + \frac{16}{2}$

$= 4 + 8$

$= 12$ local max

9-5 cont.

2. a) $f(x) = x^4 - 6x^2 + 10$

$f'(x) = 4x^3 - 12x$

$0 = 4x^3 - 12x$

$0 = 4x(x^2 - 3)$

$4x = 0, \quad x^2 - 3 = 0$

$x = 0 \qquad x^2 = 3$

$x = \pm\sqrt{3}$

$f''(x) = 12x^2 - 12$

$f(0) = 12 \cdot 0^2 - 12$

$= 0 - 12$

$= -12$ neg.

$f(\sqrt{3}) = 12\sqrt{3}^2 - 12$

$= 36 - 12$

$= 24$ pos

$f(-\sqrt{3}) = 12(-\sqrt{3})^2 - 12$

$= 36 - 12$

$= 24$ pos

$f(0) = 0^4 - 6 \cdot 0^2 + 10$

$= 0 - 0 + 10$

$= 10$ local max.

$f(\sqrt{3}) = \sqrt{3}^4 - 6\sqrt{3}^2 + 10$

$= 9 - 18 + 10$

$= 1$ local min

$f(-\sqrt{3}) = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 + 10$

$= 9 - 18 + 10$

$= 1$ local min

b) $g(x) = \dfrac{x}{(2x-3)^2}$

$= x(2x-3)^{-2}$

$g'(x) = x(-2)(2x-3)^{-3}(2) + 1(2x-3)^{-2}$

$= \dfrac{-4x}{(2x-3)^3} + \dfrac{1}{(2x-3)^2}$

$= \dfrac{-4x}{(2x-3)^3} + \dfrac{2x-3}{(2x-3)^3}$

$= \dfrac{-4x+2x-3}{(2x-3)^3}$

$= \dfrac{-2x-3}{(2x-3)^3}$

$0 = \dfrac{-2x-3}{(2x-3)^3}$

$0 = -2x - 3$

$3 = -2x$

$\dfrac{-3}{2} = x$

$g(-\tfrac{3}{2}) = \dfrac{-\frac{3}{2}}{[2(-\frac{3}{2})-3]^2}$

$= \dfrac{-\frac{3}{2}}{(-6)^2}$

$= \dfrac{-\frac{3}{2}}{36}$

$= \dfrac{-3}{2} \cdot \dfrac{1}{36} = \dfrac{-3}{72} = \dfrac{-1}{24}$ local min

$g'(x) = \dfrac{(-2x-3)}{(2x-3)^3}$

$= (-2x-3)(2x-3)^{-3}$

$g''(x) = (-2x-3)(-3)(2x-3)^{-4}(2) + -2(2x-3)^{-3}$

$= \dfrac{-6(-2x-3)}{(2x-3)^4} + \dfrac{-2}{(2x-3)^3}$

$g''(-\tfrac{3}{2}) = \dfrac{-6(-2(-\frac{3}{2})-3)}{(2(-\frac{3}{2})-3)^4} + \dfrac{-2}{(2(-\frac{3}{2})-3)^3}$

$= \dfrac{-6 \cdot 0}{(-6)^4} + \dfrac{-2}{(-6)^3}$

$= 0 + \dfrac{-2}{-216}$

$= \dfrac{1}{108}$ pos

9-cont.

2. c) $f(t) = \frac{t^2}{2t+5}$

$= t^2(2t+5)^{-1}$

$f'(t) = t^2(-1)(2t+5)^{-2}(2) + 2t(2t+5)^{-1}$

$= -2t^2(2t+5)^{-2} + 2t(2t+5)^{-1}$

$= \frac{-2t^2}{(2t+5)^2} + \frac{2t}{2t+5}$

$= \frac{-2t^2}{(2t+5)^2} + \frac{2t(2t+5)}{(2t+5)^2}$

$= \frac{-2t^2+4t^2+10t}{(2t+5)^2}$

$= \frac{2t^2+10t}{(2t+5)^2}$

$0 = \frac{2t^2+10t}{(2t+5)^2}$

$0 = 2t^2+10t$

$0 = 2t(t+5)$

$2t=0, \quad t+5=0$

$t=0 \quad t=-5$

$f''(t) = \frac{(2t+5)^2(4t+10) - (2t^2+10t)2(2t+5)(2)}{[(2t+5)^2]^2}$

$= \frac{(2t+5)^2(2)(2t+5) - 4(2t)(t+5)(2t+5)}{(2t+5)^4}$

$= \frac{\cancel{(2t+5)}[2(2t+5)^2 - 8t(t+5)]}{(2t+5)^{\cancel{4}\,3}}$

$= \frac{2(4t^2+20t+25) - 8t^2-40t}{(2t+5)^3}$

$= \frac{\cancel{8t^2}+\cancel{40t}+50-\cancel{8t^2}-\cancel{40t}}{(2t+5)^3}$

$= \frac{50}{(2t+5)^3}$

9-5 cont.

2 c) cont.

$f''(0) = \frac{50}{(2\cdot 0+5)^3}$

$= \frac{50}{5^3}$

$= \frac{\cancel{50}\,2}{\cancel{125}\,5}$

$= \frac{2}{5}$ pos.

$f''(-5) = \frac{50}{[2(-5)+5]^3}$

$= \frac{50}{(-10+5)^3}$

$= \frac{50}{(-5)^3}$

$= \frac{50}{-125} = -\frac{2}{5}$ neg

$f(0) = \frac{0^2}{2\cdot 0+5}$

$= 0$ local min

$f(-5) = \frac{(-5)^2}{2(-5)+5}$

$= \frac{25}{-10+5}$

$= \frac{25}{-5} = -5$ local max

3. a) $y = x - x^3$

$y' = 1 - 3x^2$

$0 = 1 - 3x^2$

$3x^2 = 1$

$x^2 = \frac{1}{3}$

$x = \pm\frac{1}{\sqrt{3}}$

$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}}\right)^3$

$= \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}$

$= \frac{3}{3\sqrt{3}} - \frac{1}{3\sqrt{3}}$

$= \frac{2}{3\sqrt{3}}$ local max

$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} - \left(-\frac{1}{\sqrt{3}}\right)^3$

$= -\frac{1}{\sqrt{3}} - \left(-\frac{1}{3\sqrt{3}}\right)$

$= -\frac{3}{3\sqrt{3}} + \frac{1}{3\sqrt{3}}$

$= -\frac{2}{3\sqrt{3}}$ local min

$y'' = -6x$

$f''\left(\frac{1}{\sqrt{3}}\right) = -6\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{-6}{\sqrt{3}}$ neg

$f''\left(-\frac{1}{\sqrt{3}}\right) = -6\left(-\frac{1}{\sqrt{3}}\right)$

$= \frac{6}{\sqrt{3}}$ pos.

$\pm\frac{2}{3\sqrt{3}} = \pm 0.385$

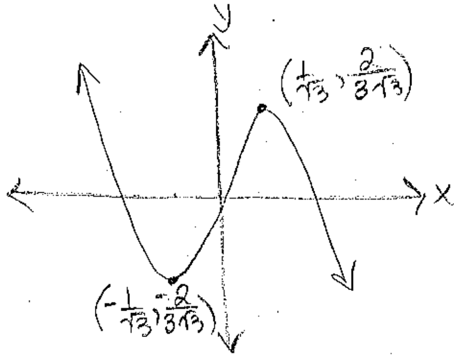
$\pm\frac{1}{\sqrt{3}} = \pm 0.577$

9-5 cont.

3. a) cont.

$-6x > 0$ Concave up for $x > 0$

$x > 0$ Concave down for $x < 0$



b) $y = 3x^5 - 25x^3 + 60x$

$y' = 15x^4 - 75x^2 + 60$

$0 = 15x^4 - 75x^2 + 60$

$0 = 15(x^4 - 5x^2 + 4)$

$0 = 15(x^2 - 4)(x^2 - 1)$

$0 = 15(x-2)(x+2)(x-1)(x+1)$

$x = 2, -2, 1, -1$

$f(2) = 3 \cdot 2^5 - 25 \cdot 2^3 + 60 \cdot 2$

$= 96 - 200 + 120$

$= 16$ local min

$f(-2) = 3(-2)^5 - 25(-2)^3 + 60(-2)$

$= -96 + 200 - 180$

$= -16$ local max

$f(1) = 3 \cdot 1^5 - 25 \cdot 1^3 + 60 \cdot 1$

$= 3 - 25 + 60$

$= 38$ local min.

$f(-1) = 3(-1)^5 - 25(-1)^3 + 60(-1)$

$= -3 + 25 - 60$

$= -38$ local max

$y'' = 60x^3 - 150x$

$f''(2) = 60 \cdot 2^3 - 150 \cdot 2$

$= 480 - 300$

$= 180$ pos.

$f''(-2) = 60(-2)^3 - 150(-2)$

$= -480 + 300$

$= -180$ neg.

$f''(1) = 60 \cdot 1^3 - 150 \cdot 1$

$= 60 - 150$

$= -90$ neg

$f''(-1) = 60(-1)^3 - 150(-1)$

$= -60 + 150$

$= 90$ pos.

9-5 cont.

3. b) cont.

$60x^3 - 150x = 0$

$30x(2x^2 - 5) = 0$

$30x = 0 \; ; \; 2x^2 - 5 = 0$

$x = 0 \qquad 2x^2 = 5$

$x^2 = \frac{5}{2}$

$x = \pm\sqrt{\frac{5}{2}}$

	30X	$2X^2-5$	y''	result
$x < -\sqrt{\frac{5}{2}}$	−	−	+	concave up
$-\sqrt{\frac{5}{2}} < x < 0$	−	−	+	concave up
$0 < x < \sqrt{\frac{5}{2}}$	+	−	−	concave dn
$x > \sqrt{\frac{5}{2}}$	+	+	+	concave up

$f(0) = 3(0)^5 - 25(0)^3 + 60(0)$

$= 0$

$(0,0)$

$f(-\sqrt{\frac{5}{2}}) = 3(-\sqrt{\frac{5}{2}})^5 - 25(-\sqrt{\frac{5}{2}})^3 + 60(-\sqrt{\frac{5}{2}})$

$= -29.65 + 98.82 - 94.87$

$= -25.7$

$f(\sqrt{\frac{5}{2}}) = 3(\sqrt{\frac{5}{2}})^5 - 25(\sqrt{\frac{5}{2}})^3 + 60(\sqrt{\frac{5}{2}})$

$= 29.65 - 98.82 + 94.87$

$= 25.7$

$\sqrt{\frac{5}{2}} = 1.58$